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A NEW DEFINITION OF THE GENERAL ABELIAN LINEAR GROUP*

BY

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1. WE may give a striking definition of the general Abelian group, making use of the fruitful conception of the “compounds of a given linear homogeneous group,” introduced in recent papers by the writer.† In § 3 we determine the multiplicity of the isomorphism of a given linear homogeneous group to its compound groups. This result is applied in § 4 to show the simple relation of the Abelian group to the general linear homogeneous group in the same number of variables. In § 5 it is shown that the simple groups of composite order which are derived from the decompositions of the quaternary Abelian group and the quinary orthogonal group, each in the $GF[p^n]$, $p > 2$, are simply isomorphic. The investigation affords a proof of the simple isomorphism between the corresponding ten-parameter projective groups without the consideration of their infinitesimal transformations.

2. It will be convenient to introduce a notation more compact than that usually employed‡ for the substitutions of the general Abelian group A_{2m, p^n} on $2m$

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† *Concerning a linear homogeneous group in $C_{m, q}$ variables isomorphic to the general linear homogeneous group in m variables*, Bulletin, Dec., 1898.

The structure of certain linear groups with quadratic invariants, Proceedings of the London Mathematical Society, vol. 30, pp. 70-98.

‡ DICKSON, *The Structure of the Hypoabelian Groups*, Bulletin, pp. 495-510, July, 1898; *A Triply Infinite System of Simple Groups*, The Quarterly Journal of Pure and Applied Mathematics, pp. 169-178, 1897; JORDAN, *Traité des Substitutions*, pp. 171-179, for the case $n = 1$.

Instead of considering the cogredient linear substitutions leaving invariant (up to a factor α) the usual bilinear function it is convenient to consider here the substitutions A leaving invariant the function

$$\psi \equiv \sum_{j=1}^m \begin{vmatrix} \xi_{2j-1} & \xi_{2j} \\ \xi_{2j-1}^n & \xi_{2j}^n \end{vmatrix}$$

The conditions that A shall leave ψ invariant are seen to be (1). The *hyperabelian* group of linear homogeneous substitutions in the $GF[p^{2n}]$ on $2m$ indices which leave ψ invariant has been studied by the writer in an article presented to the London Mathematical Society.

indices in the Galois field of order p^n . The conditions that a substitution

$$A: \quad \hat{\zeta}'_i = \sum_{j=1}^{2m} a_{ij} \hat{\zeta}_j \quad (i=1, \dots, 2m)$$

shall be Abelian are the following:

$$(1) \quad \sum_{j=1}^m \begin{vmatrix} a_{2j-1\ i} & a_{2j-1\ k} \\ a_{2j\ i} & a_{2j\ k} \end{vmatrix} = a \varepsilon_{ik} \quad (i, k=1, \dots, 2m; i < k)$$

where a is a parameter $\neq 0$ depending upon the particular substitution A and where every $\varepsilon_{ik} = 0$ unless $k = i + 1 = \text{even}$, when

$$\varepsilon_{2l-1\ 2l} = 1 \quad (l=1, \dots, m).$$

The second compound $C_{2m, 2}$ of the $2m$ -ary group A_{2m, p^n} is formed by the substitutions

$$(2) \quad Y'_{i_1 i_2} = \sum_{j_1 < j_2}^{j_1, j_2=1 \dots 2m} \begin{vmatrix} a_{i_1 j_1} & a_{i_1 j_2} \\ a_{i_2 j_1} & a_{i_2 j_2} \end{vmatrix} Y_{j_1 j_2} \quad \left(i_1, i_2=1, \dots, 2m \atop i_1 < i_2 \right).$$

We readily verify that the group $C_{2m, 2}$ has the relative invariant

$$Z \equiv \sum_{i=1}^m Y_{2i-1\ 2i}.$$

Indeed, in virtue of the relations (1), we have, on applying to Z the substitution (2),

$$\begin{aligned} \sum_{i=1}^m Y'_{2i-1\ 2i} &= \sum_{j_1, j_2} \left\{ \sum_{i=1}^m \begin{vmatrix} a_{2i-1\ j_1} & a_{2i-1\ j_2} \\ a_{2i\ j_1} & a_{2i\ j_2} \end{vmatrix} \right\} Y_{j_1 j_2} \\ &= a \sum_{j_1, j_2} \varepsilon_{j_1 j_2} Y_{j_1 j_2} = a \sum_{i=1}^m Y_{2i-1\ 2i}. \end{aligned}$$

Inversely, if the substitution (2) multiply the function Z by a constant a , the relations (1) hold true. We have proved the result:

THEOREM.—*The general Abelian group A_{2m, p^n} is the largest $2m$ -ary linear homogeneous group whose second compound has as a relative invariant the linear function Z .*

3. To establish the more important theorem of § 4, we determine the multiplicity of the isomorphism of a given m -ary linear homogeneous group G_m to its q^{th} compound $C_{m, q}$, supposing that $q < m$. To the substitution

$$(\alpha_{ij}) \quad (i, j=1, \dots, m)$$

of G_m there corresponds the following substitution of $C_{m, q}$:*

* We employ Sylvester's umbral notation for determinants.

$$[a]_q: \quad Y'_{i_1 \dots i_q} = \sum_{\substack{j_1 \dots j_q \\ j_1 < j_2 < \dots < j_q}} \begin{vmatrix} i_1 & \dots & i_q \\ j_1 & \dots & j_q \end{vmatrix} Y_{j_1 \dots j_q} \quad \left(\begin{smallmatrix} i_1, \dots, i_q = 1, \dots, m \\ i_1 < i_2 < \dots < i_q \end{smallmatrix} \right).$$

Let j be an integer such that $q < j \equiv m$. Consider the matrix J of certain coefficients of the substitution $[a]_q$, viz.,

$$\begin{pmatrix} \begin{vmatrix} 1 & 2 & \dots & q-1 & q \\ 1 & 2 & \dots & q-1 & q \end{vmatrix} & - \begin{vmatrix} 1 & 2 & \dots & q-1 & q \\ 1 & 2 & \dots & q-1 & j \end{vmatrix} \dots (-1)^q & \begin{vmatrix} 1 & 2 & \dots & q-1 & q \\ 2 & 3 & \dots & q & j \end{vmatrix} \\ - \begin{vmatrix} 1 & 2 & \dots & q-1 & j \\ 1 & 2 & \dots & q-1 & q \end{vmatrix} & \begin{vmatrix} 1 & 2 & \dots & q-1 & j \\ 1 & 2 & \dots & q-1 & j \end{vmatrix} \dots (-1)^{q+1} & \begin{vmatrix} 1 & 2 & \dots & q-1 & j \\ 2 & 3 & \dots & q & j \end{vmatrix} \\ \dots & \dots & \dots \\ (-1)^q \begin{vmatrix} 2 & 3 & \dots & q \\ 1 & 2 & \dots & q-1 & q \end{vmatrix} & (-1)^{q+1} \begin{vmatrix} 2 & 3 & \dots & q \\ 1 & 2 & \dots & q-1 & j \end{vmatrix} \dots & \begin{vmatrix} 2 & 3 & \dots & q & j \\ 2 & 3 & \dots & q & j \end{vmatrix} \end{pmatrix}.$$

Consider also the matrix A of determinant Δ ,

$$A \equiv \begin{bmatrix} a_{jj} & a_{qj} \dots a_{1j} \\ a_{jq} & a_{qq} \dots a_{1q} \\ \dots & \dots \\ a_{j1} & a_{q1} \dots a_{11} \end{bmatrix}.$$

The composition of the matrices J and A gives the result

$$JA \equiv \begin{bmatrix} \Delta & 0 \dots 0 \\ 0 & \Delta \dots 0 \\ \dots & \dots \\ 0 & 0 \dots \Delta \end{bmatrix}.$$

We seek those substitutions of G_m which correspond to the identity in $C_{m,q}$. Suppose, therefore, that $[a]_q$ reduces to the identical substitution, so that the matrix J is the identity.

In this case we have

$$\Delta^{q+1} = \Delta, \quad a_{ik} = 0, \quad a_{ii} = \Delta \quad (i, k = 1, 2, \dots, q, j; i \neq k).$$

Taking in turn $j = q + 1, q + 2, \dots, m$, we have the result

$$(a_{ij}) \equiv \begin{bmatrix} \Delta & 0 \dots 0 \\ 0 & \Delta \dots 0 \\ \dots & \dots \\ 0 & 0 \dots \Delta \end{bmatrix}.$$

Hence $\Delta \neq 0$ and therefore $\Delta^q = 1$. Inversely, every such substitution of G_m corresponds to the identity in $C_{m,q}$.

THEOREM.—*The continuous group of all m -ary linear homogeneous substitutions is $(q, 1)$ fold isomorphic to its q^{th} compound ($q < m$).*

For linear substitutions in the $GF[p^n]$, we have

$$\Delta^q = 1, \quad \Delta^{p^n-1} = 1.$$

Thus we have the analogous.

THEOREM.—*The group of all m -ary linear homogeneous substitutions in the $GF[p^n]$ is $(g, 1)$ fold isomorphic to its q^{th} compound, g being the greatest common divisor of q and $p^n - 1$.*

4. From the results of §§ 2-3, we derive immediately the

THEOREM.—*According as $p = 2$ or $p > 2$, the general Abelian group A_{2m, p^n} is holodrically or hemiedrically* isomorphic to that subgroup of the second compound of the general $2m$ -ary linear homogeneous group in the $GF[p^n]$ which has as a relative invariant the linear function Z .*

The writer has shown (Bulletin, l. c.) that this second compound leaves invariant the Pfaffian

$$F_{2m} \equiv [1, 2, \dots, 2m].$$

Hence A_{2m, p^n} is isomorphic to a linear homogeneous group in $m(2m - 1)$ variables X_{ij} with coefficients in the $GF[p^n]$ and having as relative invariants the functions

$$F_{2m}, \quad Z \equiv \sum_{i=1}^m X_{2i-1, 2i}.$$

To the subgroup* of the latter which leaves these functions absolutely invariant there corresponds a self-conjugate subgroup of A_{2m, p^n} , which leaves the customary bilinear function absolutely invariant. This subgroup, containing only substitutions of determinant ± 1 , may be designated as the *special* Abelian group A'_{2m, p^n} . It has the self-conjugate substitution which changes the signs of all the $2m$ indices. Except for $(2m, p^n) = (2, 2), (2, 3)$ and $(4, 2)$, the quotient group H_{2m, p^n} is simple.†

5. THEOREM.—*For $p > 2$, the simple group H_{4, p^n} , having the order*

* Since the Abelian group contains the substitution $\xi'_i = -\xi_i (i = 1, 2, \dots, 2m)$.

† This group has been studied by the writer in the Proceedings of the London Mathematical Society, l. c., §§ 22-33.

‡ DICKSON, *A triply infinite system of simple groups*, The Quarterly Journal of Mathematics, July, 1897; *ibid.*, April, 1899, for the cases $p = 2, 2m = 4, n > 1$, previously unconsidered.

$$\frac{1}{2}(p^{4n} - 1)(p^{2n} - 1)p^{4n},$$

is simply isomorphic to the simple subgroup of equal order of the quinary orthogonal group in the $GF[p^n]$.

On introducing the invariant $Z \equiv Y_{12} + Y_{34}$ as a new variable in place of Y_{34} , the general substitution [see (2)] of the second compound of A'_{4,p^n} becomes, for $p > 2$:

	$Y_{12} - \frac{1}{2}Z$	Y_{13}	Y_{14}	Y_{23}	Y_{24}
$(Y_{12} - \frac{1}{2}Z)' =$	$2 \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} - 1$	$\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$	$\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix}$	$\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}$	$\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}$
$Y_{13}' =$	$2 \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix}$	$\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix}$	$\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix}$	$\begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix}$	$\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}$
$Y_{14}' =$	$2 \begin{vmatrix} 1 & 4 \\ 1 & 2 \end{vmatrix}$	$\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix}$	$\begin{vmatrix} 1 & 4 \\ 1 & 4 \end{vmatrix}$	$\begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix}$	$\begin{vmatrix} 1 & 4 \\ 2 & 4 \end{vmatrix}$
$Y_{23}' =$	$2 \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}$	$\begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix}$	$\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix}$	$\begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix}$	$\begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix}$
$Y_{24}' =$	$2 \begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix}$	$\begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix}$	$\begin{vmatrix} 2 & 4 \\ 1 & 4 \end{vmatrix}$	$\begin{vmatrix} 2 & 4 \\ 2 & 3 \end{vmatrix}$	$\begin{vmatrix} 2 & 4 \\ 2 & 4 \end{vmatrix}$

It is therefore a substitution on five indices leaving absolutely invariant the function

$$\varphi \equiv (\tfrac{1}{2}Z)^2 - [1234] \equiv (Y_{12} - \tfrac{1}{2}Z)^2 + Y_{13}Y_{24} - Y_{14}Y_{23}.$$

This second compound is simply isomorphic to H_{4,p^n} . Indeed, the former is hemiedrically isomorphic to A'_{4,p^n} by §3; while to the substitution changing the sign of every index there corresponds the identity in the second compound.

By a simple transformation of indices* the function φ can be given the form

$$\sum_{i=1}^5 x_i^2.$$

Hence the second compound is simply isomorphic to a subgroup O_{5,p^n} of the total orthogonal group O of determinant unity. From the result of §16 of the paper in the Proceedings of the London Mathematical Society, above cited, it follows that O_{5,p^n} does not contain the substitution which extends the simple subgroup of O of order

* See the first pages of the article, *Determination of the structure of all linear homogeneous groups in a Galois field which are defined by a quadratic invariant*, American Journal of Mathematics, July, 1899.

$$\frac{1}{2}(p^{4n} - 1)(p^{2n} - 1)p^{4n}$$

to the total group O . Hence O_{5, p^n} is this simple subgroup.

This investigation also proves the theorem due to Lie: *The projective group of a linear complex in space of three dimensions is isomorphic to the projective group of a non-degenerate surface of the second order in space of four dimensions, each group having ten parameters.*

UNIVERSITY OF CALIFORNIA, August 20, 1898.
